

MINIMAL LAMINATIONS WITH PRESCRIBED CONVEX CURVATURE BLOWUP

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ABSTRACT. We construct minimal laminations with prescribed singularities on a line segment using perturbation techniques and PDE methods. In addition to the singular set, the rate of curvature blowup is also prescribable in our construction, and we show that all curvature blowup rates between quadratic and quartic arise. Our result generalizes an earlier result of the author and of Hoffman and White.

1. INTRODUCTION

In this article, we use PDE methods to construct embedded minimal disks in a fixed ball containing a straight line segment, and with prescribed values for $|A|^2$ along the line segment, subject to a very general constraint:

Theorem 1. *Let \mathcal{H} denote the helicoid and let $\lambda(\sigma) : [0, 1] \rightarrow R^+$ denote a positive “scale function” satisfying*

$$(1) \quad \lambda < C_0, \quad |\dot{\lambda}| \leq C_1 \lambda^\epsilon, \quad |\ddot{\lambda}| \leq C_1 \lambda^\epsilon, \quad |\ddot{\lambda} \lambda^2| \leq C_1 \lambda^\epsilon,$$

for constants $\epsilon > 0$, $C_0 > 0$ and $C_1 > 0$, and where “ \cdot ” denotes derivation σ . Then given ϵ and C_1 , there is C_0 and $R_0 > 0$ so that for all functions satisfying the conditions above, there is a surface \mathcal{H}^ such that:*

- (1) \mathcal{H}^* is embedded in the cylinder $\{x^2 + y^2 \leq R_0^2, 0 \leq z \leq 1\}$.
- (2) \mathcal{H}^* is locally a scaled small perturbation of \mathcal{H} in the following sense: There is $c > 0$ so that for $\sigma \in [0, 1]$, the surface

$$\tilde{\mathcal{H}}_\sigma^* := \lambda^{-1}(\sigma) (\mathcal{H}^* - (0, 0, \sigma)) \cap B(0, c/\lambda(\sigma))$$

is a smooth perturbation of $\mathcal{H} \cap B(0, \epsilon/\lambda(p))$.

Our Theorem recovers the main results of [HW] and [K1], which were in the former case obtained using geometric measure theory and in the latter the Weierstrass Representation, and which we state in plain language below:

Corollary 1 (From [HW] and [K1]). *Given a compact subset K of a line segment in R^3 , there exists a sequence of minimal disks, properly embedded in a ball B containing K , converging to a minimal lamination with singularities exactly on K .*

The simplest case of Theorem 1 arises when the subset K coincides with the line segment itself, in which case a sequence of such embedded minimal disks is realized by helicoids of decreasing scale. Several simpler, non-trivial cases precede Theorem 1, including the main theorems of [Kh], [D] and the original result of Colding-Minicozzi in [CM4], all of which used the Weierstrass Representation.

One of the motivations for approaching the problem of constructing embedded minimal disks and their limit laminations with PDE techniques is their great flexibility and direct control on the geometry of the resulting surfaces they provide, at least relative to the Weierstrass Representation. For example, we are able to construct, for each compact set K as in the statement of Theorem 1, smooth families of minimal laminations singularities exactly on K and in particular we show that all super-quadratic curvature blowup rates arise among such laminations:

Corollary 2. *For each $\epsilon > 0$, there exists an embedded minimal disk Σ , embedded in the half cylinder $\{x^2 + y^2 \geq 1, z \geq 0\}$ in R^3 , and so that*

$$(1/C)h^{-2(1+\epsilon)} \leq \sup_{z \geq h} |A_\Sigma|^2 \leq Ch^{-2(1+\epsilon)}$$

The case $\epsilon = 0$ was treated in [BK1], in which a minimal lamination of the exterior of a positive cone was exhibited. A theorem of Meeks, Perez and Ros ([MPR]) gives that no such lamination laminates a ball containing the singularity. The statement of Corollary 2 above can be strengthened slightly to say that minimal surfaces described arise as limits of properly embedded minimal disks.

Unlike the Weierstrass representation techniques of [Kh], [D], [K1], and [CM4], the PDE techniques employed here should have several straightforward generalizations. Firstly, there seems to be no major obstruction to proving an analogue of Theorem 1 in which the straight line is replaced by an arbitrary smooth curve. Secondly, our techniques seem to apply more or less directly to singly periodic minimal surfaces other than the helicoid. In particular, it seems quite possible that applying our technique to Scherk towers would yield some extremely pathological examples of embedded minimal surfaces; namely, embedded in a punctured ball, with infinite topology and with no smooth extension to the whole ball.

The precise regularity of the curve required for such a construction to work is no greater than a C^3 requirement, although we have not systematically addressed this question in this article. Work of Colding-Minicozzi in [CM1], [CM2], [CM3] famously shows that K must be contained in a Lipschitz curve, and more recently B. White in [W1] has strengthened the regularity to C^1 . Meeks and Weber in [MW] have shown that every $C^{1,1}$ curve arises as the blowup set. J. Bernstein and G. Tinaglia have considered problems relating limit to laminations in [BT]. Constructions related to the present have been undertaken in [BK1] and [BK2].

2. PRELIMINARIES

2.1. Basic notation and conventions. Throughout this paper we make extensive use of cut-off functions, and we adopt the following notation: Let $\psi_0 : R \rightarrow [0, 1]$ be a smooth function such that

- (1) ψ_0 is non-decreasing
- (2) $\psi_0 \equiv 1$ on $[1, \infty)$ and $\psi_0 \equiv 0$ on $(-\infty, -1]$
- (3) $\psi_0 - 1/2$ is an odd function.

For $a, b \in R$ with $a \neq b$, let $\psi[a, b] : R \rightarrow [0, 1]$ be defined by $\psi[a, b] = \psi_0 \circ L_{a,b}$ where $L_{a,b} : R \rightarrow R$ is a linear function with $L(a) = -3, L(b) = 3$. Then $\psi[a, b]$ has the following properties:

- (1) $\psi[a, b]$ is weakly monotone.
- (2) $\psi[a, b] = 1$ on a neighborhood of b and $\psi[a, b] = 0$ on a neighborhood of a .
- (3) $\psi[a, b] + \psi[b, a] = 1$ on R .

2.2. Norms and Hölder spaces.

Definition 1. *Given a function $u \in C^{j,\alpha}(D)$, where $D \subset R^m$, the (j, α) localized Hölder norm is given by*

$$\|u\|_{j,\alpha}(p) := \|u : C^{j,\alpha}(D \cap B_1(p))\|.$$

We let $C_{loc}^{j,\alpha}(D)$ denote the space of functions for which $\| - \|_{j,\alpha}$ is pointwise finite.

Definition 2. *Given a positive function $f : D \rightarrow R$, we let the space $C^{j,\alpha}(D, f)$ be the space of functions for which the weighted norm $\| - : C^{j,\alpha}(D, f) \|$ is finite, where we take*

$$\|u : C^{j,\alpha}(D, f)\| := \sup_{p \in D} f(p)^{-1} \|u\|_{j,\alpha}(p)$$

Definition 3. *Let \mathcal{X} and \mathcal{Y} be two Banach spaces with norms $\| - : \mathcal{X} \|$ and $\| - : \mathcal{Y} \|$, respectively. Then $\mathcal{X} \cap \mathcal{Y}$ is naturally a Banach space with norm $\| - : \mathcal{X} \cap \mathcal{Y} \|$ given by*

$$\|f : \mathcal{X} \cap \mathcal{Y}\| = \|f : \mathcal{X}\| + \|f : \mathcal{Y}\|.$$

Let \mathcal{X} be a Banach space with norm $\|-\| : \mathcal{X}$ and suppose $S \subset \mathcal{X}$. For convenience, throughout the paper we will sometimes write $\|-\| : S$, where for any $f \in S$ we simply let

$$\|f : S\| := \|f : \mathcal{X}\|.$$

2.3. Estimating homogeneous quantities. In this section we record a formalized procedure for producing estimates for quantities defined on immersions that scale homogeneously. The quantities and results which we record have already appeared in [BK2], and for this reason we omit the proofs and instead refer the reader to [BK2] for details.

Let E be the Euclidean space $E := E^{(1)} \times E^{(2)} = R^{3 \times 2} \times R^{3 \times 4}$. We denote points of E by $\underline{\nabla} = (\nabla, \nabla^2)$, where

$$\nabla = (\nabla_1, \nabla_2) \quad \nabla^2 = (\nabla_{11}^2, \nabla_{22}^2, \nabla_{12}^2, \nabla_{21}^2).$$

We then consider functions $\Phi(\underline{\nabla})$ on E with the property

$$\Phi(c\underline{\nabla}) = c^d \Phi(\underline{\nabla})$$

for real numbers c and d . We call such a function a *homogeneous function of degree d* . A homogeneous degree d function has the property that its j^{th} derivative $D^{(j)}\Phi$ is homogeneous degree $d - j$.

Notice E is just a Euclidean space so for any $V \in E$, we make the identification $T_V E = E$. We extend this for each $k \in \mathbb{Z}^+$ and observe that $D^{(k)}\Phi(V) : E^k \rightarrow R$. For clarity we provide the following definition.

Definition 4. Let $k \in \mathbb{Z}^+$, $V, W_1, \dots, W_k \in E$. Then

$$D^{(k)}\Phi \Big|_V (W_1 \otimes \dots \otimes W_k) := D^{(k)}\Phi(V)(W_1 \otimes \dots \otimes W_k).$$

For brevity, we denote the k -th tensor product of W with itself by

$$\otimes^{(k)} W := W \otimes \dots \otimes W.$$

Definition 5. Given an immersion $\phi : D \subset R^2 \rightarrow R^3$, we set $\underline{\nabla}[\phi] := (\nabla\phi, \nabla^2\phi)$. A homogeneous quantity of degree d on ϕ is then a function of the form $\Phi[\phi] := \Phi(\underline{\nabla}[\phi])$ for some homogeneous function Φ on E .

We refer to a map $\underline{\nabla}(s, z) : D \subset R^2 \rightarrow E$ as an *immersion* if the quantity

$$(2) \quad \mathbf{a}(\underline{\nabla}) = \mathbf{a}(\nabla) := 2\sqrt{\det \nabla^T \nabla} / |\nabla|^2$$

is everywhere non-zero, and otherwise we refer to it simply as a *vector field*.

Definition 6. Given an immersion $\underline{\nabla}$ and a vector field \mathcal{E} , we set

$$(3) \quad R_{\Phi, \mathcal{E}}^{(k)}(\underline{\nabla}) := \int_0^1 \frac{(1-\sigma)^k}{k!} D\Phi^{(k+1)} \Big|_{\underline{\nabla}(\sigma)} (\otimes^{(k+1)} \mathcal{E}) d\sigma$$

where $\otimes^{(k)} \mathcal{E}$ denotes the k -fold tensor product of \mathcal{E} with itself and where $\underline{\nabla}(\sigma) := \underline{\nabla} + \sigma \mathcal{E}$.

When $\underline{\nabla}$ and \mathcal{E} are of the form $\underline{\nabla} = \underline{\nabla}\phi$ and $\mathcal{E} = \underline{\nabla}V$ we write

$$R_{\Phi, V}^{(k)}(\phi) := R_{\Phi, \mathcal{E}}^{(k)}(\underline{\nabla}).$$

Note that $R_{\Phi, \mathcal{E}}(\underline{\nabla})$ is simply the order k Taylor remainder so that:

Proposition 1. We have

$$(4) \quad \Phi(\underline{\nabla} + \mathcal{E}) - \Phi(\underline{\nabla}) - D\Phi \Big|_{\underline{\nabla}}(\mathcal{E}) - \dots - \frac{1}{k!} D^{(k)}\Phi \Big|_{\underline{\nabla}} (\otimes^{(k)} \mathcal{E}) = R_{\Phi, \mathcal{E}}^{(k)}(\underline{\nabla})$$

Proposition 2. There exists $\tilde{\epsilon} > 0$ such that if $\underline{\nabla} : D \rightarrow E$ is an immersion and $\mathcal{E} : D \rightarrow E$ is a vector field satisfying

$$\|\mathcal{E} : C^{j, \alpha}(D, \mathbf{a}(\nabla)|\nabla|)\| \leq C(j, \alpha)\tilde{\epsilon}, \quad \text{and} \quad \ell_{j, \alpha}(\nabla) := \|\nabla : C^{j, \alpha}(D, |\nabla|)\| < \infty,$$

then

$$\left\| R_{\Phi, \mathcal{E}}^{(k)}(\underline{\nabla}) : C^{j, \alpha}(D, |\nabla|^d) \right\| \leq C(\Phi, \ell_{j, \alpha}, \mathbf{a}, k) \|\mathcal{E} : C^{j, \alpha}(D, |\nabla|)\|^{k+1}.$$

3. OUTLINE

We fix a positive function $\lambda(\sigma) : (0, 1) \rightarrow R$ satisfying the conditions of Theorem 1 and set

$$(5) \quad z(\sigma) := \int_0^\sigma \frac{d\sigma}{\lambda(\sigma)}.$$

Let $\sigma(z)$ denote the inverse of $z(\sigma)$. Then for any object $\Phi(\sigma)$ depending on the parameter σ , we will throughout simply write

$$\Phi(z) := \Phi \circ \sigma(z).$$

We will also use “ \cdot ” to denote derivation in σ and “ $'$ ” to denote the derivative in z . We observe that

$$\Phi'(z) = \lambda(z)\dot{\Phi}(z).$$

The map

$$B(x, y, z) = \sigma(z)e_z + \lambda(z)xe_x + \lambda(z)ye_y$$

is then easily seen to be a diffeomorphism of R^3 and we set

$$(6) \quad F^*(s, z) := B \circ F(s, z)$$

where $F(s, z)$ is the conformal parametrization of the helicoid given by

$$(7) \quad F(s, z) := \cosh(s) \sin(z)e_x + \cosh(s) \cos(z)e_y + e_z.$$

Our goal is then to find a graph over G which gives an embedded minimal surface as in Theorem 1. As a first step we need to estimate various geometric quantities on G including the mean curvature, metric, and the stability operator

Proposition 3. *Let g^* , ν^* , A^* , $|A^*|$, Δ^* , \mathcal{L} and H denote respectively the metric, unit normal, second fundamental form, the length of the second fundamental form, Laplace-Beltrami operator, and the stability operator and the mean curvature of F^* . Then:*

(1) *With $g^* = g_{ss}ds^2 + g_{zz}dz^2 + 2g_{sz}dsdz$, we have*

$$g_{ss} = \lambda^2 \cosh^2(s), \quad g_{zz} = \lambda^2 \left(\cosh^2(s) + \dot{\lambda}^2 \sinh^2(s) \right), \quad g_{sz} = \dot{\lambda} \lambda^2 \sinh(s) \cosh(s)$$

(2) *We have that*

$$\nu^* = -\cosh^{-1}(s)e_r(x) + \tanh(s)e_z,$$

where we have set $e_r(x) = \sin(x)e_x + \cos(x)e_y$.

(3) *With $A^* = A_{ss}ds^2 + A_{zz}dz^2 + 2A_{sz}dsdz$ we have*

$$A_{ss} = 0, \quad A_{zz} - \dot{\lambda}\lambda \tanh(s), \quad A_{sz} = -\lambda.$$

(4) *It holds that*

$$\begin{aligned} \lambda^2 \cosh^2(s) \Delta^* &= (1 + \dot{\lambda}^2 \tanh^2(s)) \partial_{ss} + \partial_{zz} - 2\dot{\lambda} \tanh(s) \partial_{sz} \\ &\quad + \left\{ 2\dot{\lambda}^2 \tanh(s) \cosh^{-2}(s) - \ddot{\lambda} \lambda \tanh(s) \right\} \partial_s - \dot{\lambda} \cosh^{-2}(s) \partial_z. \end{aligned}$$

(5) *It holds that*

$$(8) \quad |A^*|^2 = \lambda^{-2} \cosh^{-4}(s) \left(2 + 2\dot{\lambda}^2 \tanh^2(s) \right).$$

(6) Set

$$\tilde{\mathcal{L}} = \Delta + 2 \cosh^{-2}(s)$$

where $\Delta = \partial_{ss}^2 + \partial_{zz}^2$ is the flat laplacian on R^2 . Then we can write

$$\lambda^2 \cosh^2(s) \mathcal{L}^* = \tilde{\mathcal{L}} + \mathcal{E},$$

where \mathcal{E} is the operator given explicitly by

$$\begin{aligned} \mathcal{E}(s, z) &= 2\dot{\lambda}^2 \sinh^2(s) + \dot{\lambda}^2 \tanh^2(s) \partial_{ss} - 2\dot{\lambda} \tanh(s) \partial_{sz} \\ &\quad + \left\{ 2\dot{\lambda}^2 \tanh(s) \cosh^{-2}(s) - \ddot{\lambda} \lambda \tanh(s) \right\} \partial_s - \dot{\lambda} \cosh^{-2}(s) \partial_z. \end{aligned}$$

$$(7) \quad H^* = -\dot{\lambda} \lambda \tanh(s) \cosh^{-2}(s).$$

Definition 7. Given a function $u : R^2 \rightarrow R$ we set

$$F^*[u](s, z) := F^*(s, z) + \lambda(z)u(s, z)\nu^*(s, z)$$

From this and Proposition 3 we get the immediate corollary:

Proposition 4. Let $u : R^2 \rightarrow R$ be a locally class $C^{2,\alpha}$ function on R^2 and assume that

$$(9) \quad \|u : C^{2,\alpha}(R^2, \cosh(s))\| \leq \epsilon.$$

Then for $\epsilon > 0$ sufficiently small, $F^*[u]$ is a class $C^{2,\alpha}$ immersion and it holds that

$$(10) \quad \tilde{H}^*[u] := \lambda(z) \cosh^2(s) H[F^*[u]] = \dot{\lambda} \tanh(s) + \tilde{\mathcal{L}}u + \tilde{R}^*[u]$$

where R^* satisfies the estimate

$$\|R^*[u]\|_{0,\alpha} \leq C \left(\dot{\lambda} \|u\|_{2,\alpha} + \|u\|_{2,\alpha}^2 \right)$$

Once Proposition 4 is established, it remains to study the linear problem for $\tilde{\mathcal{L}}$:

$$(11) \quad \tilde{\mathcal{L}}u = \dot{\lambda} \tanh(s).$$

Most of the inhomogeneous term in (11) can be integrated directly: It is easily checked that $\tanh(s)$ is in the kernel of

$$(12) \quad \tilde{L} := \partial_{ss}^2 + 2 \cosh^{-2}(s).$$

Variation of parameters then gives that

$$(13) \quad \tilde{L}^{-1}(f) := \left(\int_0^s \tanh^{-2}(s') \int_0^{s'} \tanh(s'') f(s'') ds'' ds' \right) \tanh(s)$$

is an inverse for \tilde{L} that preserves zero dirichlet condition at $s = 0$. Set:

$$(14) \quad u_0(s, z) := \tilde{L}^{-1}(\tanh(s))$$

Then we have directly:

Proposition 5. The following statements hold:

(1) It holds that

$$\tilde{\mathcal{L}}(\dot{\lambda} u_0) = \dot{\lambda}(z) \tanh(s) + \left(\ddot{\lambda} \lambda^2 + \ddot{\lambda}^2 \lambda \right) z u_0 := \dot{\lambda}(z) \tanh(s) + E,$$

where E is determined implicitly.

(2) The function u_0 satisfies the estimate:

$$\|u\|_{j,\alpha}(s) \leq 1 + s^2.$$

Thus, we are left with solving the linear problem for the remainder term E . Using the convexity assumption (1) we then have directly that

$$(15) \quad \|E : C^{0,\alpha}(R^2, \lambda^\epsilon(1+s^2))\| \leq CC_1.$$

Since the s parameter is approximately the logarithmic radial distance of $\lambda^{-1}(z)F^*(z, s)$ from the z axis, the domain of our graph must contain at least the set $\{(s, z) : |s| \leq \log(c/\lambda(z))\}$ if we wish to obtain minimal surfaces in tubes of fixed radius independent of λ . We in fact include much more and find solutions on the domain

$$(16) \quad \Lambda := \{(s, z) : |s| \leq \underline{\ell}(z)\}, \quad \underline{\ell}(z) := \lambda^{-\tau\epsilon}(z)$$

where τ is a small positive constant to be determined. Then with $\epsilon_0 := (1 - 2\tau)\epsilon$, the definition of Λ and the estimate for E in (15) give

$$(17) \quad \|E : C^{0,\alpha}(\Lambda, \lambda^{\epsilon_0})\| := \gamma < \infty.$$

Our main invertibility statement is then:

Proposition 6. *Let $E : \Lambda \rightarrow R$ be a locally class $C^{0,\alpha}$ function satisfying the estimate in (17). Then there is a function $u : \Lambda \rightarrow R$ such that*

(1) *It holds that*

$$\tilde{\mathcal{L}}u = E$$

(2) *Set $\epsilon_2 = (1 - 20\tau)\epsilon_0$. Then u satisfies the estimate:*

$$\|u : C^{2,\alpha}(\Lambda, \lambda^{\epsilon_1})\| \leq C\|E : C^{0,\alpha}(\Lambda, \lambda^{\epsilon_0})\|$$

The primary difficulty in proving Theorem 6 has to do with interactions of regions of Λ corresponding to different scales. If one assumes that the ratio

$$\lambda_{\min}/\lambda_{\max}$$

is uniformly bounded below, then Proposition 6 becomes significantly easier. We essential prove Proposition 6 as a corollary to Proposition 7 below, which considers inhomogeneous terms supported on strips of a fixed height and satisfying a strong orthogonality condition.

Definition 8. *The domain $\Lambda(a, b)$ is given as follows:*

$$\Lambda(a, b) := \{|z| \leq b, |s| \leq a\}.$$

Proposition 7. *Let E be a class $C^{0,\alpha}$ function supported on $\Lambda(\ell, 2\pi)$ and satisfying*

$$\int_0^\ell E(s, z) \tanh(s) ds = 0.$$

Then, given $N > 2\pi$ there is function $u_N : \Lambda(\ell, N) \rightarrow R$ such that

(1) *It holds that*

$$\tilde{\mathcal{L}}u_N := E.$$

(2) *For $|z| \leq N$, $u(s, z)$ satisfies the Robin boundary condition*

$$\partial_s u_N(\ell, z) \tanh(\ell) - u_N(\ell, z) / \cosh^2(\ell) = 0.$$

(3) *u_N satisfies the Neuman boundary condition*

$$\partial_z u_N(s, \pm N) = 0.$$

An immediate consequence of Proposition 7 is that the orthogonality condition on E is inherited by the solution u_N . This implies a one dimensional Poincare inequality for u_N along lines that gives uniform control on the solutions u_N in N .

Proposition 8. *There is a universal constant C so that the function u_N in the statement of Proposition 7 satisfies the following estimate:*

$$\|u_N : L^1(\Lambda(\ell, N))\|, \quad \|\partial_s u_N : L^1(\Lambda(\ell, N))\| \leq C\ell^2 \|E : C^{0,\alpha}(\Lambda(\ell, 2\pi), 1)\|.$$

From this, it then immediately follows that the functions u_N are uniformly bounded in N and converge smoothly as $N \rightarrow \infty$ to a limiting function u_∞ solving the linear problem for E on the domain $\Lambda(\ell) := \Lambda(\ell, \infty)$. This and a maximum principle implies the weighted Hölder estimate for u_∞ :

Proposition 9. *There is a constant C so that*

$$\left\| u_\infty : C^{2,\alpha} \left(\Lambda(\ell), \frac{1}{1+|z|} \right) \right\| \leq C \ell^3 \|E : C^{0,\alpha}(\Lambda(\ell, 2\pi), 1)\|.$$

Proposition 6 then follows by applying Proposition 9 to pieces of the inhomogeneous term that lie in strips of fixed width, summing the resulting solutions, and using the convexity of the scale function λ to show that only regions of comparable scale interact. The existence of the minimal graph over F^* defined on Λ then follows almost immediately. We formulate its existence in terms of a fixed point of the mapping Ψ on the ball

$$(18) \quad \Xi := \{u \in C_{\text{loc}}^{2,\alpha}(\Lambda) : \|u : C^{2,\alpha}(\Lambda, \lambda^{\epsilon_1})\| \leq \zeta\},$$

where Ψ is given by

$$(19) \quad \Psi(u) := u - \tilde{\mathcal{L}}^{-1} \tilde{H}^* u,$$

where ζ is a constant to be determined, and where $\tilde{\mathcal{L}}^{-1}$ denotes the inverse to $\tilde{\mathcal{L}}$ between the weighted Hölder spaces described in Proposition 6.

Proposition 10. *For λ sufficiently small, the function $\Psi(u)$ is defined on the set Ξ and acts as a contraction.*

4. PROOF OF PROPOSITION 3 AND COROLLARY 4

To prove Proposition 3 we first record the first and second derivatives of F^* .

Lemma 1. *We have*

$$(20) \quad \begin{aligned} F_s^* &= \lambda \cosh(s) \mathbf{e}_r, & F_z^* &= \lambda' \sinh(s) \mathbf{e}_r + \lambda \sinh(s) \mathbf{e}_r' + \lambda \mathbf{e}_z \\ F_{ss}^* &= \lambda \sinh(s) \mathbf{e}_r, & F_{sz}^* &= \lambda' \cosh(s) \mathbf{e}_r + \lambda \cosh(s) \mathbf{e}_r', \\ F_{zz}^* &= \lambda'' \sinh(s) \mathbf{e}_r - \lambda \sinh(s) \mathbf{e}_r + 2\lambda' \sinh(s) \mathbf{e}_r' + \lambda' \mathbf{e}_z. \end{aligned}$$

4.1. The unit normal.

Lemma 2. *The unit normal of F^* is*

$$(21) \quad \nu(s, z) = -\cosh^{-1}(s) \mathbf{e}_r' + \tanh(s) \mathbf{e}_z$$

Proof. This follows immediately as

$$\begin{aligned} F_s^* \wedge F_z^* &= (\lambda(z) \cosh(s) \mathbf{e}_r) \wedge (\lambda(z) \sinh(s) \mathbf{e}_r + e^{\delta z} \sinh(s) \mathbf{e}_r' + \lambda(z) \mathbf{e}_z) \\ &= \lambda^2 \sinh(s) \mathbf{e}_r \wedge \mathbf{e}_r' + \lambda^2 \cosh(s) \mathbf{e}_r \wedge \mathbf{e}_z \\ &= \lambda^2 \cosh(s) \sinh(s) \mathbf{e}_z - \lambda^2 \cosh(s) \mathbf{e}_r' \end{aligned}$$

and thus

$$|F_s^* \wedge F_z^*|^2 = \lambda^4 \cosh^4(s).$$

□

4.2. The metric.

Lemma 3. *Let $g^* = g_{ss}^* ds^2 + g_{zz}^* dz^2 + 2g_{sz}^* ds dz$ be the metric of g^* . Then*

$$g_{ss}^* = \lambda^2 \cosh^2(s), \quad g_{zz}^* = \lambda^2 \left(\cosh^2(s) + \dot{\lambda}^2 \sinh^2(s) \right), \quad g_{sz}^* = \dot{\lambda} \lambda^2 \sinh(s) \cosh(s).$$

Proof. This follows directly from (20). □

As a direct consequence,

$$(22) \quad |g^*| := \det g^* = \lambda^4 \cosh^4(s)$$

and the components of the dual metric are

$$(23) \quad F^{*ss} = \lambda^{-2} \cosh^{-2}(s)(1 + \dot{\lambda}^2 \tanh^2(s)), \quad F^{*zz} = \lambda^{-2} \cosh^{-2}(s), \quad F^{*sz} = -\dot{\lambda} \lambda^{-2} \tanh(s) \cosh^{-2}(s).$$

4.3. The second fundamental form.

Lemma 4. *Let $A := A_{ss}ds^2 + A_{zz}dz^2 + 2A_{sz}dsdz$ be the second fundamental form, and let $|A|^2$ be its length. Then we have*

$$A_{ss} = 0, \quad A_{zz} = -\dot{\lambda} \lambda \tanh(s), \quad A_{sz} = -\lambda$$

and

$$|A|^2 = \lambda^{-2} \cosh^{-4}(s) \left(2 + 2\dot{\lambda}^2 \tanh^2(s) \right).$$

Proof. We determine the components of the second fundamental form by using (20) and (21). To obtain the expression for the length of the second fundamental form, we write

$$\begin{aligned} |A|^2 &= \begin{pmatrix} A_{sz}A_{sz} & A_{sz}A_{zs} & A_{sz}A_{zz} \\ A_{zs}A_{sz} & A_{zs}A_{zs} & A_{zs}A_{zz} \\ A_{zz}A_{sz} & A_{zz}A_{zs} & A_{zz}A_{zz} \end{pmatrix} * \begin{pmatrix} g^{ss}g^{zz} & g^{sz}g^{zs} & g^{sz}g^{zz} \\ g^{zs}g^{sz} & g^{zz}g^{ss} & g^{zz}g^{sz} \\ g^{zs}g^{zz} & g^{zz}g^{zs} & g^{zz}g^{zz} \end{pmatrix} \\ &= \lambda^2 \begin{pmatrix} 1 & 1 & \dot{\lambda} \tanh(s) \\ 1 & 1 & \dot{\lambda} \tanh(s) \\ \dot{\lambda} \tanh(s) & \dot{\lambda} \tanh(s) & \dot{\lambda}^2 \tanh^2(s) \end{pmatrix} * \frac{\lambda^{-4}}{\cosh^4(s)} \begin{pmatrix} 1 + \dot{\lambda}^2 \tanh^2(s) & \dot{\lambda}^2 \tanh^2(s) & -\dot{\lambda} \tanh(s) \\ \dot{\lambda}^2 \tanh^2(s) & 1 + \dot{\lambda}^2 \tanh^2(s) & -\dot{\lambda} \tanh(s) \\ -\dot{\lambda} \tanh(s) & -\dot{\lambda} \tanh(s) & 1 \end{pmatrix} \\ &= \cosh^{-4}(s) \lambda^{-2} \left(2 + 2\dot{\lambda}^2 \tanh^2(s) \right). \end{aligned}$$

□

Lemma 5. *Let H^* be the mean curvature of F^* . Then*

$$H^*(s, z) = -\dot{\lambda} \lambda \tanh(s) \cosh^{-2}(s).$$

4.4. The Laplace operator.

Lemma 6. *Let Δ^* be the Laplace operator on F^* . Then*

$$\begin{aligned} \lambda^2 \cosh^2(s) \Delta_g &= (1 + \dot{\lambda}^2 \tanh^2(s)) \partial_{ss} + \partial_{zz} - 2\dot{\lambda} \tanh(s) \partial_{sz} \\ &\quad + \left\{ 2\dot{\lambda}^2 \tanh(s) \cosh^{-2}(s) - \ddot{\lambda} \lambda \tanh(s) \right\} \partial_s - \dot{\lambda} \cosh^{-2}(s) \partial_z. \end{aligned}$$

Proof. This follows directly from the expressions for the coefficients of the dual metric and its determinant in (22) and (23), and the local coordinate expression for. □

Lemmas 2, 3, 4, 5 and 6 then collectively prove Proposition 3.

5. PROOF OF PROPOSITION 7, PROPOSITION 8 AND PROPOSITION 9

Proposition 7 follows from the fact that $\tanh(s)$ spans the kernel of $\tilde{\mathcal{L}}$ on $\Lambda(\ell, N)$ with the boundary conditions stated in Proposition 7 (2) and (3), which we prove below:

Proposition 11. *Let $\phi : \Lambda(\ell, N) \rightarrow R$ satisfy:*

- (1) $\tilde{\mathcal{L}}\phi = 0$.
- (2) $\partial_s \phi(\ell, z) \tanh(\ell) - \phi(\ell, z) / \cosh^2(\ell) = 0$.
- (3) $\partial_z \phi(s, \pm N) = 0$.

Then ϕ is a multiple of $\tanh(s)$.

Proof. For fixed s , let $\phi_k(s)$ denote the Fourier coefficients of $\phi(s, z)$, so that

$$\phi(s, z) = \sum \phi_k(s) e^{ikz}.$$

Then ϕ_k satisfies the equation:

$$\tilde{L}\phi_k - k^2 \phi_k = \phi_k'' + (2 \cosh^{-2}(s) - k^2) \phi_k = 0.$$

For each fixed k , we have that $\phi_k(0) = 0$ and after possibly re-normalizing, we can assume that $\phi_k(\ell) = \tanh(\ell)$. Assume also that $\phi_k'(\ell) \leq \tanh'(\ell)$. Thus, for $\ell - s$ sufficiently small and positive, we have that $\phi_k(s) > \tanh(s)$. Let s_0 be the largest real number less than ℓ so that

$$\phi_k(s_0) = \tanh(s_0).$$

Then $w(s) := \phi_k(s) - \tanh(s)$ is positive on the interior of $[s_0, \ell]$ and vanishes at the endpoints. However, we have

$$(24) \quad \tilde{L}w(s) = \tilde{L}\phi_k = k^2 \phi_k > 0$$

so that $w(s)$ cannot have an interior maximum, which gives a contradiction. Thus, we have that

$$(25) \quad \phi_k'(\ell) > \tanh'(\ell) = \cosh^{-2}(\ell).$$

It then immediately follows that if ϕ satisfies the Robin boundary condition in the statement, then only the zero mode is present in the Fourier expansion, which gives

$$\phi(s) = c \tanh(s)$$

and completes the proof. \square

From this Proposition 7 immediately follows:

proof of Proposition 7. The existence of weak solutions and their higher regularity follows from standard theory. \square

In order to prove Proposition 8, We first observe that the solutions u_N inherit the orthogonality condition from E .

Proposition 12. *It holds that*

$$(26) \quad \int_0^\ell u_N(s, z) \tanh(s) ds = 0$$

for all $z \in [-N, N]$.

Proof. We have

$$\partial_{zz}^2 u_N + \partial_{ss}^2 u_N + 2 \cosh^{-2}(s) u_N = \partial_{zz}^2 u_N + \tilde{L}u_N = E.$$

Multiplying both sides by $\tanh(s)$, integrating by parts and using the Robin boundary condition then gives

$$\int_0^\ell (\partial_{zz}^2 u_N)(s, z) \tanh(s) ds = \partial_{zz}^2 \left(\int_0^\ell u_N(s, z) \tanh(s) ds \right) = 0.$$

The boundary conditions then immediately imply $\left(\int_0^\ell u_N(s, z) \tanh(s) ds \right)$ is constant in z . The conclusion then follows by adding a multiple of $\tanh(s)$ to u_N if necessary. \square

Definition 9. For a function f belonging to the space $W^{1,2}([0, \ell])$ we set

$$e_\ell(f) := \int_0^\ell f'^2(s) ds - 2 \int_0^\ell f^2(s) \cosh^{-2}(s) ds - f^2(\ell) \tanh'(\ell) / \tanh(\ell).$$

Proposition 13 (One dimensional weighted Poincare Inequality). *There is a universal constant $\beta > 0$ independent of ℓ so that: Let $f \in W^{1,2}([0, \ell])$ satisfy the orthogonality condition:*

$$(27) \quad \int_0^\ell f(s) \tanh(s) / \cosh^2(s) ds = 0.$$

Then it holds that

$$e_\ell(f) \geq \beta \|f : L^2([0, \ell], \cosh^{-2}(s))\|^2.$$

Proof. Suppose not, and let (f_k, ℓ_k) be a sequence of functions satisfying

$$|e_{\ell_k}(f_k)| \leq 1/j \quad \|f : L^2([0, \ell_k], \cosh^{-2}(s))\|^2 = 1.$$

We then immediately get that $\{f_k\}$ is a uniformly bounded sequence in $W^{1,2}[0, \ell]$ independent of ℓ . Assuming $\ell_k \rightarrow \infty$, we then get that f_k strongly sub converges on compact subsets of $[0, \infty]$ to a limiting function f in $L^2([0, \infty], \cosh^{-2}(s))$. Moreover, the uniform energy bound on the sequence $\{f_k\}$ gives that

$$|f_k(s)| \leq Cs^{1/2}, \quad |f(s)| \leq Cs^{1/2}.$$

The dominated convergence theorem then implies that f satisfying the following conditions:

$$e(f) := e_\infty(f) = 0, \quad \|f : L^2([0, 1], \cosh^{-2}(s))\| = 1, \quad \int_0^\infty f \tanh(s) / \cosh^2(s) ds = 0.$$

The Dirichlet condition $f(0) = 0$ then implies that f is a non-trivial multiple of $\tanh(s)$, which violates that last condition above. This concludes the proof. \square

Proposition 14. *Let f be a function satisfying the hypothesis of Proposition 13. Then it holds that*

$$\left(1 - \frac{2}{2 + \beta}\right) \int_0^\ell f'^2(s) ds \leq 2e_\ell(f).$$

Proof. The inequality in Proposition 13 can be written explicitly as:

$$\int_0^\ell f'^2(s) ds - \int_0^\ell f^2(2 + \beta) \cosh^{-2}(s) + f^2(\ell) \tanh'(\ell) / \tanh(\ell) \geq 0.$$

Equivalently:

$$\left(1 - \frac{2}{2 + \beta}\right) \int_0^\ell f'^2(s) ds + \left(\frac{2}{2 + \beta} - 1\right) f^2(\ell) \tanh'(\ell) / \tanh(\ell) \leq e(f)$$

Since $|f^2(\ell)| \leq \ell \int_0^\ell f'^2(s) ds$, the claim follows directly. \square

There is a discrepancy between the orthogonality condition we would like the solutions u_N to inherit—namely, the weighted orthogonality condition in Proposition 13 (27)—and the orthogonality condition we are able to control—that recorded in Proposition 12. Nonetheless, we can show that solutions inherit enough of the weighted orthogonality condition to prove sufficient bounds.

Proposition 15. *Let $f(s)$ be a function in $L^2([0, \ell])$ satisfying the orthogonality condition (26) in Proposition 12. Then we can write*

$$(28) \quad f(s) = \alpha \tanh(s) + g(s)$$

where g satisfies the following conditions

$$\int_0^\ell g(s) \tanh(s) / \cosh^2(s) ds = 0, \quad \int_0^\ell f^2(s) ds \leq \int_0^\ell g^2(s) ds.$$

Proof. Write

$$f = \alpha \tanh(s) + (f - \alpha \tanh(s)) := \tanh(s) + g$$

Then choosing α appropriately, it is clear that we can arrange for g to be orthogonal to $\tanh(s)/\cosh^2(s)$. In particular, we choose α so that

$$(29) \quad \alpha = \frac{\int_0^\ell f(s) \tanh(s) / \cosh^2(s) ds}{\int_0^\ell \tanh^2(s) / \cosh^2(s) ds}.$$

□

Proposition 16. *Let f and g be as in the statement of Proposition 15. Then it holds that*

$$e_\ell(f) = e_\ell(g) \geq \left(1 - \frac{2}{2 + \beta}\right) \left(\int_0^\ell g'^2(s) ds\right)$$

Proof. We have that

$$e_\ell(f) = B[f, f].$$

It is then clear that the right hand side above is a bilinear form and we have

$$\begin{aligned} e_\ell(f) &= B[\alpha \tanh(s) + g, \alpha \tanh(s) + g] \\ &= \alpha^2 B[\tanh(s), \tanh(s)] + B[g, g] + 2\alpha B[g, \tanh(s)] \\ &= B[g, g] \\ &= e_\ell(g). \end{aligned}$$

where the last equality above follows from the definition of $e(-)$ and $B[-, -]$ and where the third line follows from the second since $\tanh(s)$ is in the kernel of \tilde{L} with the boundary conditions (2) and (3). □

We are now ready to prove Proposition 8:

Proof of Proposition 8. In the following, in order to conveniently reference Propositions 13, 14 and 16 while using their notational conventions, we set

$$f(s, z) = u_N(s, z) = \alpha(z) \tanh(s) + g(s, z).$$

where the multiple α is chosen so that

$$\int_0^{2\ell} g(s, z) \tanh(s) ds = 0.$$

Additionally, we will throughout the proof abbreviate $\Lambda := \Lambda(\ell, N)$, $\Lambda_0 := \Lambda(\ell, 2\pi)$ and we will use “ ’ ” to denote derivation with respect to s . Multiplying the left hand side of (1) by f , integrating in s from 0 to ℓ and using the boundary conditions gives:

$$\begin{aligned} I(z) &:= \int_0^\ell f(s, z) \partial_{zz}^2 f(s, z) ds - \int_0^\ell (\partial_s f)^2 ds + 2 \int_0^\ell f^2(s, z) \cosh^{-2}(s) ds \\ &\quad + f^2(\ell, z) \tanh'(\ell) / \tanh(\ell) \\ &= \int_0^\ell f(s, z) \partial_{zz}^2 f(s, z) ds - e_\ell(f(z, -)). \end{aligned}$$

where the energy $e_\ell(-)$ is defined in Definition 9. Recall that from Proposition 16 we have $e_\ell(g(z, -)) = e_\ell(f(z, -))$. Proposition 14 then gives:

$$\begin{aligned} \left(1 - \frac{2}{2+\beta}\right) \int_{\Lambda} g'^2 &\leq \left| \int_{\Lambda_0} f E \right| \\ &\leq \left(\int_{\Lambda_0} f^2 \right)^{1/2} \left(\int_{\Lambda_0} E^2 \right)^{1/2} \\ &\leq \left(\int_{\Lambda_0} g^2 \right)^{1/2} \left(\int_{\Lambda_0} E^2 \right)^{1/2} \\ &\leq C\ell \left(\int_{\Lambda_0} g'^2 \right)^{1/2} \left(\int_{\Lambda_0} E^2 \right)^{1/2}. \end{aligned}$$

Summarizing the above estimates:

$$\int_{\Lambda} g'^2 \leq C\ell^2 \int_{\Lambda_0} E^2.$$

We then have

$$\int_{\Lambda} f^2 ds \leq \int_{\Lambda} g^2 ds \leq C\ell^2 \int_{\Lambda} g'^2 \leq C\ell^4 \int_{\Lambda} E^2.$$

Applying Hölder's inequality to the above inequality then gives:

$$\left(\int_{\Lambda} |f| \right)^2 \leq \ell \int_{\Lambda} f^2 ds \leq C\ell^5 \int_{\Lambda} E^2.$$

Recalling the expression for α in (29), we then conclude that

$$|\alpha| \leq C \int_0^\ell |f| \leq C\ell^{3/2} \left(\int_{\Lambda} E^2 \right)^{1/2}.$$

It then follows that

$$\|f' : L^1\| \leq \alpha \|\cosh^{-2}(s) : L^1\| + \|g' : L^1\| \leq C\ell^3 \|E : C^{0,\alpha}(\Lambda_0, 1)\|.$$

This concludes the proof. □

The L^2 estimate for the solutions u_N in Proposition 8 and a maximum principle then immediately translate into existence function u_∞ satisfying the weighted estimate in Proposition 9.

Proposition 17. *There is a function $u_\infty : \Lambda(\ell, \infty) \rightarrow R$ such that*

- (1) *For any compact subset $K \subset \Lambda(\ell, \infty)$, the functions u_N converge to u_∞ smoothly in $C^{2,\alpha}(K)$.*
- (2) *The function u_∞ satisfies the boundary value problem:*

$$\tilde{\mathcal{L}}u_\infty = E, \quad u_\infty(0, z) = 0, \quad u'_\infty(\ell, z) \tanh(\ell) - u(\ell, z)/\cosh^{-2}(\ell) = 0.$$

- (3) *u_∞ also satisfies the orthogonality condition*

$$\int_0^\ell u_\infty(s, z) \tanh(s) ds = 0, \quad \forall z \in R.$$

- (4) *u_∞ satisfies the estimate*

$$\|u_\infty : L^1(\Lambda(\ell, \infty))\|, \quad \|\partial_s u_\infty : L^1(\Lambda(\ell, \infty))\| \leq C\ell^3 \|E : C^{0,\alpha}(\Lambda(\ell, 2\pi), 1)\|.$$

Proof. This is a direct consequence of the uniform L^2 estimates recorded for the functions u_N in Proposition 8 and standard techniques. □

We record the maximum principle below:

Proposition 18. *The function*

$$\bar{S}(w) := \sup_{(s,z) \in \Lambda(\ell, \infty), z \geq w} |u_\infty(s, z)|$$

is monotonically decreasing for $w > 2\pi$.

Proof. Suppose there is $(s_0, z_0) \in \Lambda(\ell, \infty)$ so that $u_\infty(s, z)$ obtains a positive maximum on $\Lambda^+ := \Lambda(\ell, \infty) \cap \{z \geq w\}$ and assume that (s_0, z_0) is in the interior of Λ^+ . Then there is a multiple m of $\tanh(s)$ and p in the interior of Λ^+ so that $g(s, z) := m \tanh(s) - u_\infty(s, z)$ is positive on $\Lambda^+ \setminus \{p\}$ and so that $g(p) = 0$, which gives a contradiction. Thus, (s_0, z_0) must be on the boundary of Λ^+ . Suppose now that $(s_0, z_0) = (\ell, z_0)$ for $z_0 > w$. We again choose m so that

$$g(s, z) := m \tanh(s) - u_\infty(s, z)$$

has the property that

$$g(\ell, z_0) = 0, \quad g(\ell, z) \geq 0.$$

Observe that g satisfies the Robin boundary condition (2) at $s = \ell$ and we have

$$\Delta g + 2 \cosh^{-2}(s)g = g_{zz} + g_{ss} + 2 \cosh^{-2}(s)g = 0.$$

At (ℓ, z_0) we then get

$$g_s \geq 0, \quad g_{ss} < 0.$$

Thus, g has an interior minimum at some point $p \in \Lambda^+$ and is non-negative on the boundary of Λ^+ , which gives a contradiction. We then conclude that if $\bar{S}(w)$ is finite, then it holds that

$$\bar{S}(w) = |u_\infty(s_w, w)|$$

for some $s_0 \in [0, \ell]$, from which it follows that $\bar{S}(w)$ is either monotonically decreasing or increasing in $|w|$. To see that $\bar{S}(w)$ is monotonically decreasing in $|w|$, observe that

$$\bar{S}(w) = |u_\infty(s_w, w)| \leq \int_0^\ell |u'_\infty(s, w)| ds$$

Integrating in w then gives that

$$(30) \quad \int_{-\infty}^\infty \bar{S}(w) \leq \int_{\Lambda(\ell)} |u'_\infty| \leq C\ell^3 \|E : C^{0,\alpha}(\Lambda(\ell, 2\pi), 1)\|.$$

This completes the proof of Proposition 18. \square

We can now prove Proposition 9:

Proof of Proposition 9. In the proof of Proposition 18 above, we saw that the function $\bar{S}(w)$ is an integrable function of w on $(-\infty, \infty)$, which is monotonic on the sets $(-\infty, 2\pi)$ and $(2\pi, \infty)$. It then immediately follows that $\bar{S}(w)$ satisfies the bound

$$|\bar{S}(w)| \leq C/(1 + |w|).$$

In fact, a stronger estimates hold, but this will suffice for our purposes. \square

6. PROOF OF PROPOSITION 6

Proposition 6 is a direct consequence of Proposition 9 and the convex pinching property of the scale functions.

Proposition 19. *There is a constant A so that, given $z_0 \in [0, 1]$, then for all $z \in [0, 1]$ with*

$$|z - z_0| \leq A\lambda^\epsilon(z_0)$$

it holds that

$$(31) \quad \left| \frac{\lambda(z)}{\lambda(z_0)} - 1 \right| \leq 1/2.$$

Proposition 20. For $\sigma_0 \in (0, 1)$, let $\delta = \delta(\sigma_0)$ be maximal so that: For $\sigma \in (0, 1)$ satisfying $|\sigma - \sigma_0| \leq \delta$ it holds that

$$\left| \frac{\lambda(\sigma)}{\lambda(\sigma_0)} - 1 \right| \leq 1/2.$$

Then we have the following estimate:

$$\delta(\sigma_0) \geq 3\lambda^{1-\epsilon}(\sigma_0).$$

Proof. For σ as in the statement of the proposition, we have

$$\begin{aligned} |\lambda(\sigma) - \lambda(\sigma_0)| &= \left| \int_{\sigma_0}^{\sigma} \dot{\lambda}(\sigma') d\sigma' \right| \\ &\leq \delta \sup_{\sigma} |\dot{\lambda}(\sigma)| \\ &\leq 3/2\delta(\lambda(\sigma_0))^{\epsilon}, \end{aligned}$$

where in the last line above we have used the convex pinching assumptions in (1). With $\sigma = \sigma_0 + \delta$ and using the maximality of δ we get that

$$|\lambda(\sigma) - \lambda(\sigma_0)| = \lambda(\sigma_0)/2 \leq 3/2\delta\lambda^{\epsilon}(\sigma_0).$$

It then follows directly that

$$\delta \geq 3\lambda^{1-\epsilon}(\sigma_0).$$

This completes the proof. \square

Proof of Proposition 19. Let σ_0 and $\delta = \delta(\sigma_0)$ be as in Proposition 20 and set $z_0 = z(\sigma_0)$, $z = z(\sigma)$. Then for $|\sigma - \sigma_0| \leq \delta$ we have from Proposition 20 that

$$\begin{aligned} \left| \frac{\lambda(z)}{\lambda(z_0)} - 1 \right| &\leq 1/2. \\ z - z_0 &= \int_{\sigma_0}^{\sigma} \frac{d\sigma'}{\lambda(\sigma')} = \frac{1}{\lambda(\sigma_0)} \int_{\sigma_0}^{\sigma} \frac{\lambda(\sigma_0)}{\lambda(\sigma')} d\sigma' \geq 2/3 \frac{|\sigma - \sigma_0|}{\lambda(\sigma_0)} \end{aligned}$$

with $\sigma = \sigma_0 + \delta$, we then get

$$\begin{aligned} |z - z_0| &\geq 2/3 \frac{\delta}{\lambda(\sigma_0)} \\ &\geq 2/3 \frac{\lambda^{1-\epsilon}(\sigma_0)}{\lambda(\sigma_0)} \\ &\geq 2/3 \lambda^{\epsilon}(\sigma_0). \end{aligned}$$

This completes the proof. \square

Definition 10. We set

$$z_j := j\pi, \quad \lambda_j := \lambda(z_j), \quad \ell_j := \underline{\ell}(z_j) \quad j \in \mathbb{N}.$$

Additionally, we let Λ_j be the domain given as follows:

$$\Lambda_j := \{(s, z) : |z| \leq 2\pi, |s| \leq 2\ell_j\}.$$

We also let $\{\psi_j(z)\}$ be a smooth partition of unity on R such that

(1) It holds that

$$\psi_j(z + \pi) = \psi_{j+1}(z)$$

(2) $\psi_0(z)$ is an even function of z .

(3) The support of ψ_0 is contained in the interval $[-3/2\pi, 3/2\pi]$.

Proposition 21. Set $E_j(s, z) := \psi_j(z)E(s, z)$. Then there is a function $v_j : \Lambda_j \rightarrow R$ so that

(1) It holds that

$$\tilde{\mathcal{L}}v_j = E_j.$$

- (2) *There is universal constant K (depending only of the $C^{2,\alpha}$ norm of ψ_0 and the invertibility constant in Proposition 9) so that: The function v_j satisfies the weighted Holder estimate:*

$$\left\| v_j : C^{2,\alpha} \left(\Lambda(2\ell_j), \frac{1}{1 + |z - z_j|} \right) \right\| \leq K \lambda_j^{\epsilon_1},$$

where $\epsilon_1 := (1 - 10\tau)\epsilon_0$.

Before proving Proposition 21, we observe a few facts.

Proposition 22. *It holds that*

$$\tilde{L}(s \tanh(s) - 1) = 0.$$

Definition 11. *Let $\varphi(s) : \mathbb{R} \rightarrow \mathbb{R}$ denote a fixed smooth function such that*

- (1) *For $s \geq 2$ it holds that $\varphi(s) = s \tanh(s) - 1$.*
- (2) *$\varphi(s)$ is an odd function of s : $\varphi(s) = -\varphi(-s)$.*

Proposition 23. *Let $\psi(z)$ be fixed smooth cutoff function with support on the interval $[-2\pi, 2\pi]$ and set*

$$\bar{f}(s, z) = \psi(z)\varphi(s), \quad \bar{g}(s, z) = z\psi(z)\varphi(s).$$

Then the following statements hold:

- (1) *For ℓ sufficiently large it holds that*

$$\left| \int_{\mathbb{R}^2} (\tilde{\mathcal{L}}\bar{f}) \tanh(s) \right| > 1/2, \quad \left| \int_{\mathbb{R}^2} (\tilde{\mathcal{L}}\bar{g}) z \tanh(s) \right| > 1/2.$$

- (2) *It holds that*

$$\int_{\mathbb{R}^2} (\tilde{\mathcal{L}}\bar{f}) z \tanh(s) = \int_{\mathbb{R}^2} (\tilde{\mathcal{L}}\bar{g}) \tanh(s) = 0.$$

Proof. Computing directly gives

$$\tilde{\mathcal{L}}\bar{f} = \psi''\varphi + \psi\tilde{L}\varphi = I + II.$$

We then have that

$$\int_{\mathbb{R}^2} I \tanh(s) = \int_0^{2\ell} \varphi(s) \tanh(s) \left(\int_{-2\pi}^{2\pi} \psi''(z) dz \right) ds = 0.$$

Additionally, we have

$$\begin{aligned} \int_{\mathbb{R}^2} II \tanh(s) &= \int_{-2\pi}^{2\pi} \psi(z) \int_0^{2\ell} (\tilde{L}\varphi) \tanh(s) ds \\ &= \int_{-2\pi}^{2\pi} \psi(z) \int_0^{2\ell} (\varphi'(2\ell) \tanh(2\ell) - \varphi(2\ell) \cosh^{-2}(2\ell)) dz. \end{aligned}$$

Similarly, we have

$$\begin{aligned} \int_{\mathbb{R}^2} (\tilde{\mathcal{L}}\bar{g}) z \tanh(s) &= \int_0^{2\ell} \partial_z \bar{g}(s, 2\pi) 2\pi \tanh(s) - \bar{g}(s, 2\pi) \tanh(s) ds - \int_0^{2\ell} \partial_z \bar{g}(s, -2\pi) (-2\pi) \tanh(s) - \bar{g}(s, -2\pi) \tanh(s) ds \\ &\quad + \int_{-2\pi}^{2\pi} \psi(z) z^2 (\varphi'(2\ell) \tanh(2\ell) - \varphi(2\ell) \cosh^{-2}(2\ell)) dz \\ &= \int_{-2\pi}^{2\pi} \psi(z) z^2 (\varphi'(2\ell) \tanh(2\ell) - \varphi(2\ell) \cosh^{-2}(2\ell)) dz \\ &> 0. \end{aligned}$$

The remaining claims are then direct consequences of the symmetries that the functions involved. \square

Proposition 24. *The following statements hold:*

- (1) The function $E(s, z_j + z)$ is supported on $\Lambda(2\ell_j, 2\pi)$.
(2) There are unique multiples \bar{a}_j and \bar{b}_j so that the function

$$\hat{E}_j(s, z) := E_j(s, z_j + z) - \bar{a}_j \left(\tilde{\mathcal{L}}\bar{f} \right)(s, z) - \bar{b}_j \left(\tilde{\mathcal{L}}\bar{g} \right)(s, z)$$

satisfies the following orthogonality conditions:

$$\int_{\Lambda(2\pi, 2\ell)} \hat{E}_j(s, z) \tanh(s) ds = \int_{\Lambda(2\ell_j, 2\pi)} \hat{E}(s, z) z \tanh(s) ds = 0.$$

- (3) The multiples \bar{a}_j and \bar{b}_j satisfy the estimates

$$|\bar{a}_j|, |\bar{b}_j| \leq C\ell_j \|E_j : C^{0,\alpha}(\Lambda(2\pi, 2\ell_j), 1)\| \leq C\ell_j \lambda_j^{\epsilon_0}.$$

- (4) It holds that

$$\begin{aligned} \|\hat{E}_j : C^{0,\alpha}(\Lambda(2\pi, 2\ell), 1)\| &\leq \|E_j : C^{0,\alpha}(\Lambda(2\pi, 2\ell), 1)\| + \bar{a}_j \|\bar{f} : C^{0,\alpha}(\Lambda(2\pi, 2\ell), 1)\| \\ &\quad + \bar{b}_j \|g : C^{0,\alpha}(\Lambda(2\pi, 2\ell), 1)\| \\ &\leq C\lambda_j^{\epsilon_0} (1 + \ell_j^2) \end{aligned}$$

Proof. (1) is clear. (2) and (3) and (4) are direct consequences of Proposition 23. \square

Proof of Proposition 21. We wish to apply Proposition 9. In order to do this directly, our error term must satisfy the strong orthogonality condition along lines stated in Proposition 7. Set

$$a(z) := \left(\int_0^{2\ell_j} \hat{E}(s, z) \tanh(s) ds \right) / \left(\int_0^{2\ell_j} \tanh^2(s) ds \right), \quad b(z) := za(z)$$

Observe that from Proposition 24 we have:

$$\int_{-2\pi}^{2\pi} a(z) dz = \int_{\Lambda(2\pi, 2\ell_j)} \hat{E}(s, z) \tanh(s) = 0, \quad \int_{-2\pi}^{2\pi} b(z) dz = \int_{\Lambda(2\pi, 2\ell_j)} \hat{E}(s, z) z \tanh(s) = 0.$$

Set

$$A(z) := \int_{-\infty}^z \int_{-\infty}^{z'} a(z'') dz'' = z \int_{-\infty}^z a(z') dz' - \int_{-\infty}^z b(z') dz'$$

where the second equality follows from integration by parts. We then put

$$h(s, z) := A(z) \tanh(s)$$

It then follows directly that $h(s, z)$ is supported on the set $\{|z| \leq 2\pi\}$ and that

$$\tilde{\mathcal{L}}h(s, z) = a(z) \tanh(s).$$

Moreover, by construction, we have that

$$\int_0^{2\ell_j} \left(\hat{E}(s, z) - a(z) \tanh(s) \right) \tanh(s) ds = \int_0^{2\ell_j} \hat{E}(s, z) \tanh(s) ds - a(z) \int_0^{2\ell_j} \tanh^2(s) ds = 0.$$

The function

$$F(s, z) := \hat{E}(s, z) - a(z) \tanh(s)$$

then satisfies the bound

$$\|F : C^{0,\alpha}(\Lambda(2\pi, 2\ell_j))\| \leq C\lambda_j^{\epsilon_0} (1 + \ell_j^2) \leq \lambda_j^{(1-4\tau)\epsilon}$$

We then apply Proposition 9 to obtain a function v satisfying:

$$\tilde{\mathcal{L}}v(s, z) = F(s, z) = E(s, z + z_j) - \bar{a} \left(\tilde{\mathcal{L}}\bar{f} \right)(s, z) - \bar{b} \left(\tilde{\mathcal{L}}\bar{g} \right)(s, z) - a(z) \tanh(s),$$

and the claimed bounds. We conclude by setting

$$v_j(s, z) := v(s, z - z_j) + \bar{a}\bar{f}(s, z - z_j) + \bar{b}\bar{g}(s, z - z_j) + A(z - z_j) \tanh(s).$$

\square

Definition 12. We let Λ_j^* be the domain given by:

$$\Lambda_j^* := \{(s, z) : |s| \leq 2\ell, |z - z_j| \leq \rho_j\}$$

where where $\rho_j := A\lambda_j^{-\epsilon}/6$. Additionally, we let $\psi_j^*(z)$ denote the cutoff function given by:

$$(32) \quad \psi_j^*(z) := \psi_0[\rho_j, \rho_j/2](|z|).$$

Proposition 25. The following statements hold:

- (1) The domain $\Lambda \cap \{|z - z_j| \leq \rho_j\}$ is contained in Λ_j^* and the boundary curve $\partial(\Lambda \cap \{|z - z_j| \leq \rho_j\})$ is strictly separated from the boundary curve $\{s = 2\ell\}$ on Λ_j^* .
- (2) The functions ψ_j^* are smooth non-negative functions with support contained in Λ_j^* and it holds that

$$|\nabla^{(k)} \psi_j^*| \leq C\rho_j^k.$$

- (3) The support of the gradient of ψ_j^* is contained in $\Lambda_j^* \setminus \Lambda_j$.

Proof. All claims in the proposition are simple consequences of Definition 12, the definition of λ in (16) and the properties of the scale function λ in (1). \square

Definition 13. The functions v_j^* and E_j^* are given as follows:

$$(33) \quad v_j^*(s, z) := \psi_j^*(z)v_j(s, z), \quad E_j^* := E_j - \tilde{\mathcal{L}}(\psi_j^*v_j)$$

Proposition 26. The following statements hold:

- (1) The supports of v_j^* and E_j^* are contained in Λ_j^* ,
- (2) Assume that z_k and z_j are such that $|z_k - z_j| \geq A\lambda_j^{-\epsilon}$, where A is as in Proposition 19. Then the supports of v_j^* and v_k^* do not intersect. Likewise, the supports of E_j^* and E_k^* do not intersect under the same conditions.
- (3) The functions v_j^* satisfy the estimates

$$\left\| v_j^* : C^{2,\alpha} \left(\Lambda_j^*, \frac{1}{1 + |z - z_j|} \right) \right\| \leq C\gamma\lambda_j^{\epsilon_1}.$$

- (4) The functions E_j^* satisfies the estimate

$$\|E_j^* : C^{0,\alpha} \left(\Lambda_j^*, \frac{\rho_j}{1 + |z - z_j|} \right)\| \leq C\gamma\lambda_j^{\epsilon_1}$$

Proof. Statement (1) is a direct consequence of the definition of ψ_j^* in Definition 12. We prove statement (2) in two cases. Let $k \in \mathbb{N}$ be such that

$$\lambda_k \leq \frac{1}{6}\lambda_j.$$

By construction, it holds that for all z belonging to the interval $(z_k - \rho_k, z_k + \rho_k)$ we have that

$$\lambda(z) \leq 3/2\lambda_k \leq 1/4\lambda_j.$$

Again, by construction we have that for z belonging to $(z_j - \rho_j, z_j + \rho_j)$, it holds that

$$\lambda(z) \geq 1/2\lambda_j.$$

Thus, if the two intervals intersect, we obtain

$$\lambda(z) \geq 2\lambda(z),$$

which is a contradiction. Now, assume that $k \in \mathbb{N}$ is such that

$$\lambda_k \geq \frac{1}{6}\lambda_j.$$

and so that z_k does not belong to the interval $(z_j - \rho_j, z_j + \rho_j)$. We then get that

$$\rho_k \leq 6^\epsilon \rho_j \leq 2\rho_j.$$

for $\epsilon > 0$ sufficiently small. Suppose now that $z \in R$ satisfies

$$|z_j - z| \leq \rho_j, \quad |z_k - z| \leq \rho_k.$$

Then from the triangle inequality we get that

$$|z_k - z_j| \leq |z_k - z| + |z_j - z| \leq \rho_k + \rho_j \leq 3\rho_j \leq (1/2)A\lambda_j^{-\epsilon}.$$

However, by assumption we have that

$$|z_k - z_j| \geq A\lambda_j^{-\epsilon}.$$

This yields a contradiction and gives the proof of Statement (2). Statements (3) and (4) follow directly for the weighted estimate for v_j in Proposition 21 (2) and the definition of ψ_j^* . \square

Proposition 27. *Set*

$$v^* := \sum_j v_j^*, \quad E^* := \sum_j E_j^*.$$

Then the following statements hold:

- (1) *The infinite sums defining v^* and E^* are locally finite and thus converge on compact subsets of Λ .*
- (2) *The function v^* satisfies the estimate*

$$\|v^* : C^{2,\alpha}(\Lambda, \lambda^{\epsilon_2})\| \leq C\gamma$$

where $\epsilon_2 = (1 - 20\tau)\epsilon$

- (3) *Given any constant $\bar{\delta} > 0$, there is C_0 in (1) so that:*

$$\|E^* : C^{0,\alpha}(\Lambda, \lambda^{\epsilon_0})\| \leq \bar{\delta}\gamma.$$

Proof. Fix a $j \in \mathbb{N}$. From Proposition 26 (2), we have on Λ_j that:

$$\begin{aligned} \left\| \sum_k v_k^* \right\|_{2,\alpha} &= \sum_{|z_k - z_j| \leq 6\rho_j} \|v_k^*\|_{2,\alpha} \\ &\leq C\gamma \sum_{|z_k - z_j| \leq 6\rho_j} \frac{\lambda_k^{\epsilon_1}}{1 + |z_k - z_j|} \\ &\leq C\gamma \lambda_j^{\epsilon_1} \int_{-6\rho_j}^{6\rho_j} \frac{1}{1 + |w|} dw \\ &\leq C\gamma \lambda_j^{\epsilon_1} \log(\rho_j) \end{aligned}$$

Setting $\epsilon_2 = (1 - 20\tau)\epsilon_0$ gives claim (2). Claim (3) follows similarly: On Λ_j^* we again have

$$\begin{aligned} \sum_k E_k^* &= \sum_{|z_k - z_j| \leq \rho_j} E_k^* \\ &\leq C\gamma \sum_{|z_k - z_j| \leq \rho_j} \frac{\rho_k \lambda_k^{\epsilon_1}}{1 + |z_k - z_j|} \\ &\leq C\gamma \rho_j \lambda_j^{\epsilon_1} \int_{-6\rho_j}^{6\rho_j} \frac{dw}{1 + w} \\ &\leq C\gamma \rho_j \lambda_j^{\epsilon_2} \end{aligned}$$

Assuming that $1 - 20\tau > 0$, we can take λ_j sufficiently small so that

$$C\lambda_j^{(1-20\tau)\epsilon} \leq \bar{\delta},$$

which gives the claim. \square

Proposition 6 now follows immediately.

Proof of Proposition 6. Let E be a function as in the statement of the proposition and set

$$\gamma := \|E : C^{0,\alpha}(\Lambda, \lambda^{\epsilon_0})\|.$$

We then apply Proposition 27 to obtain functions v^* , E^* satisfying the following estimates:

$$\|v^* : C^{2,\alpha}(\Lambda, \lambda^{\epsilon_2})\| \leq C\gamma, \quad \|E^* : C^{0,\alpha}(\Lambda, \lambda^{\epsilon_0})\| \leq \bar{\delta}\gamma.$$

We then repeat the process to obtain a sequence of functions (v_i, E_i) satisfying

$$\tilde{\mathcal{L}}v_i = E_i$$

and the following estimates:

$$\|E_i : C^{2,\alpha}(\Lambda, \lambda^{\epsilon_0})\| \leq \bar{\delta}^i \gamma, \quad \|v_i : C^{2,\alpha}(\Lambda, \lambda^{\epsilon_2})\| \leq (C\bar{\delta})^i \gamma.$$

It is then straightforward that for $\bar{\delta}$ sufficiently small, the partial sums $\sum_{i=0}^k v_i$ goes to infinity to a function v satisfying all the claims in the Proposition. This completes the proof \square

7. THE NON-LINEAR PROBLEM

Proof of Proposition 4. From Proposition (4) it follows that for λ sufficiently small we have:

$$\mathfrak{a}[F^*(s, z)] \geq 1/2, \quad |\nabla F(s, z)| > C\lambda(z) \cosh(s), \quad \ell_{j,\alpha}[F(s, z)] \leq C_0$$

and that the vector field $\mathcal{E}(s, z) := \lambda(z)u(s, z)\nu^*(s, z)$ satisfies

$$\|\mathcal{E}\|_{j,\alpha}(s, z) \leq C\lambda(z)\|u\|_{j,\alpha}(s, z) \leq C\epsilon|\nabla F(s, z)|.$$

The claim then follows directly from Proposition 2 and taking ϵ sufficiently small. \square

Proposition 28. *Let u be a function belonging to the set Ξ defined in (18). Then, given $\delta > 0$, and $\tau > 0$ satisfying $1 - 20\tau > 0$, there is C_0 sufficiently small in (1) so that: The term $R^*[u]$ defined in Proposition (4) satisfies the estimate*

$$\begin{aligned} \|R^*[u]\|_{0,\alpha}(s, z) &\leq C\zeta^2\lambda^{2\epsilon_2}\cosh^{-1}(s) + C\zeta\lambda^\epsilon\lambda^{\epsilon_1} \\ &\leq \delta\lambda^{\epsilon_0}(z). \end{aligned}$$

Proof. This is a direct consequence of the definition of Ξ in (18), Proposition 4 and the scale function assumptions in (1). \square

Theorem 2. *There is $\bar{C}_0 > 0$ and $\zeta > 0$ so that: For $C_0 \in (0, \bar{C}_0)$, The function Ψ defined in (19) is continuous and maps Ξ into Ξ and thus has a fixed point, which we denote by $u^* \in \Xi$. The surface $F^*[u]$ is then a smooth, minimal embedded surface in the ball*

$$\{(x, y, z) : x^2 + y^2 + z^2 \leq 1\}.$$

Definition 14. *We let u_1 be the function obtained by applying Proposition 6 to the function*

$$E(s, z) := -\dot{\lambda}(z)\tanh(s) + \tilde{L}u_0.$$

where u_0 is the function described in (14).

Proposition 29. *Set $v_0 := u_1 + u_0$. Then the following statements hold:*

(1) v_0 satisfies the equation:

$$\tilde{\mathcal{L}}v_0 = \tilde{H}^*[0]$$

(2) v_0 satisfies the estimate:

$$\|v_0 : C^{2,\alpha}(\Lambda, \lambda^{\epsilon_2})\| \leq C$$

Proof of Theorem 2. Proposition 4 gives that

$$\tilde{H}^*[u] = \tilde{H}^*[0] + \tilde{\mathcal{L}}u + \tilde{R}^*[u].$$

This then gives that

$$\begin{aligned} \Psi[u] - \Psi[v_0] &= u - v_0 - \tilde{\mathcal{L}}^{-1} \left(\tilde{H}^*[u] - \tilde{H}^*[v_0] \right) \\ &= \mathcal{L}^{-1} \left(\tilde{R}^*[u] - \tilde{R}^*[v_0] \right) \end{aligned}$$

We then have From Proposition 6 and Proposition 28 that

$$\|\mathcal{L}^{-1} \left(\tilde{R}^*[u] - \tilde{R}^*[v_0] \right) : C^{2,\alpha}(\Lambda, \lambda^{\epsilon_2})\| \leq C(2\delta)$$

Taking ζ sufficiently large and δ sufficiently small gives that $\Psi(\Xi) \subset (\Xi)$. Since the map Ψ depends continuously on u , the Schauder Fixed Point Theroem gives the existence of a function $u^* \in \Xi$ such that

$$\Psi(u^*) = u^*.$$

It then follows directly that $\tilde{H}^*[u] = 0$, so that $F^*[u]$ is a $C^{2,\alpha}$ minimal immersion. Higher estimates follow from standard elliptic theory. The embeddedness of $F^*[u]$ in the unit ball around the origin follows directly from the definition of Ξ . \square

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